Remarks on Stochastic Volatility Models for Equity Derivatives.

José Da Fonseca

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Objectives:

- Given an overview of some stochastic volatility models of the affine class for equity with an emphasis on implementation problems.

On the calibration of the Heston (1993) model

$$
\begin{aligned}
\frac{ds_t}{s_t} &= \sqrt{v_t}\,dw_t^1 \\
dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}\,dw_t^2 \\
dw_t^1 dw_t^2 &= \rho dt
\end{aligned}
$$

- θ is the long term vol
- κ id the mean reverting parameter
- σ is the vol-of-vol
- dw_t^1 the noise of the asset, dw_t^2 the noise of the vol

ρ controls the link between vol and asset returns

$\Downarrow$

The Skew or Leverage

Analytical and Financial properties

We do not know in closed form the density of s_T therefore:

- the computation of expectations (option prices) is a problem
- the estimation using MaxLik is a problem

but we know the MGF (Moments Generating Function) therefore

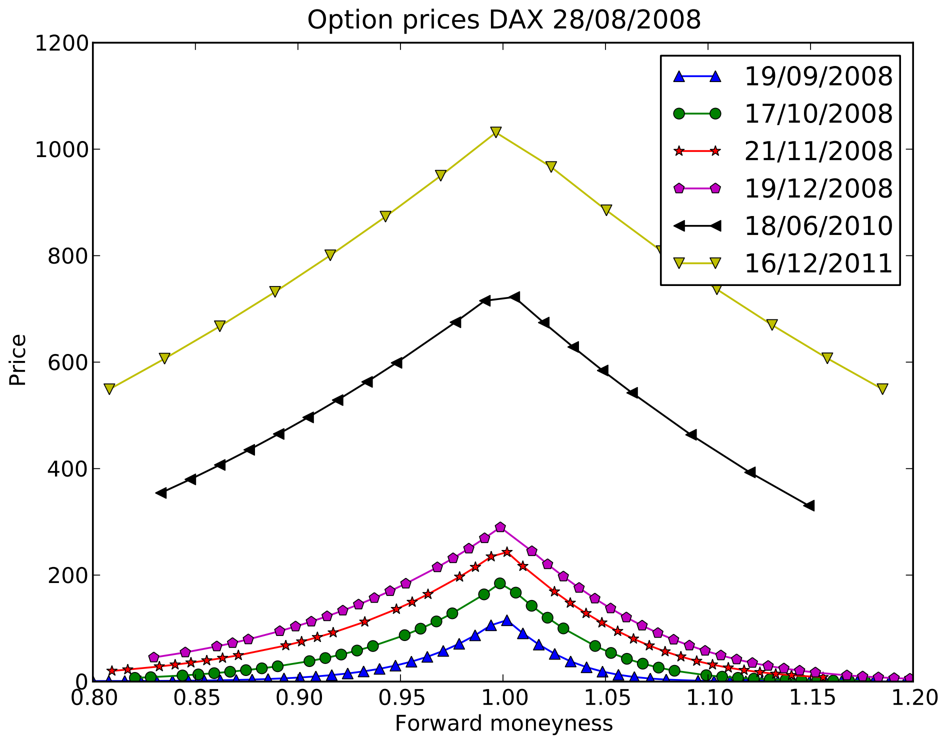
- the computation of expectations expressed as a function of the MGF is feasible (for example:option prices)
- the estimation using GMM (for example) is feasible

Quoting vanilla options

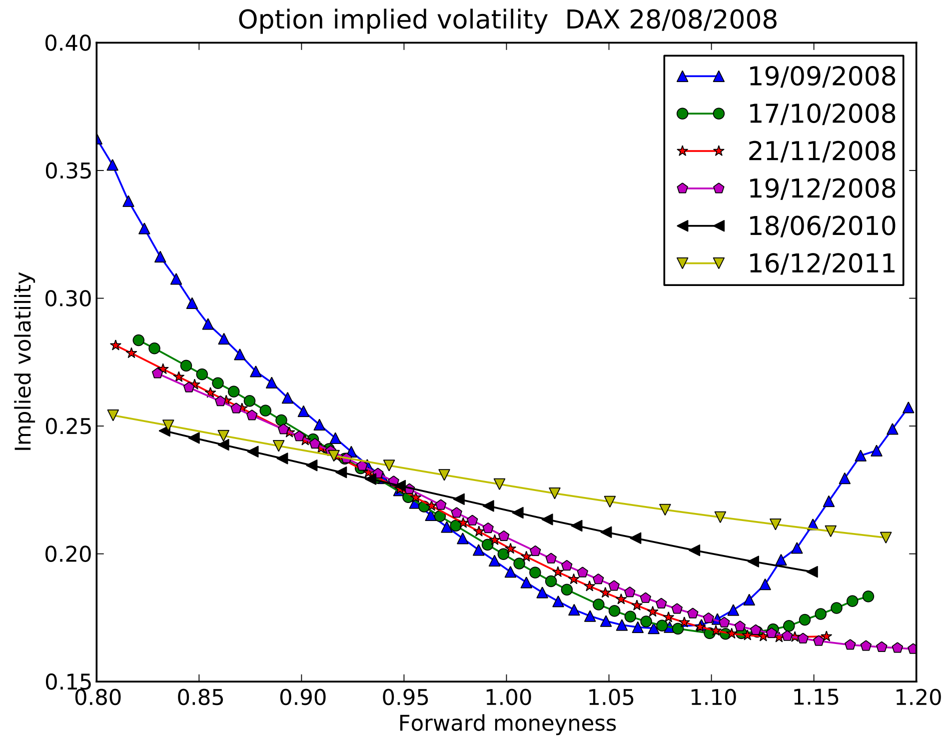
The implied volatility σ_{imp} is the quantity such that

$$\underbrace{C_{mkt}(t, T, S_t, K)}_{\text{market price}} = \underbrace{c_{bs}(t, T, S_t, K, \sigma^2_{imp}(T-t))}_{\text{price in the Black\&Scholes model}}$$

On the market we observe the “Smiles”



Option prices DAX 28/08/2008
Price
Forward moneyness
19/09/2008
17/10/2008
21/11/2008
19/12/2008
18/06/2010
16/12/2011



Option implied volatility DAX 28/08/2008
Implied volatility
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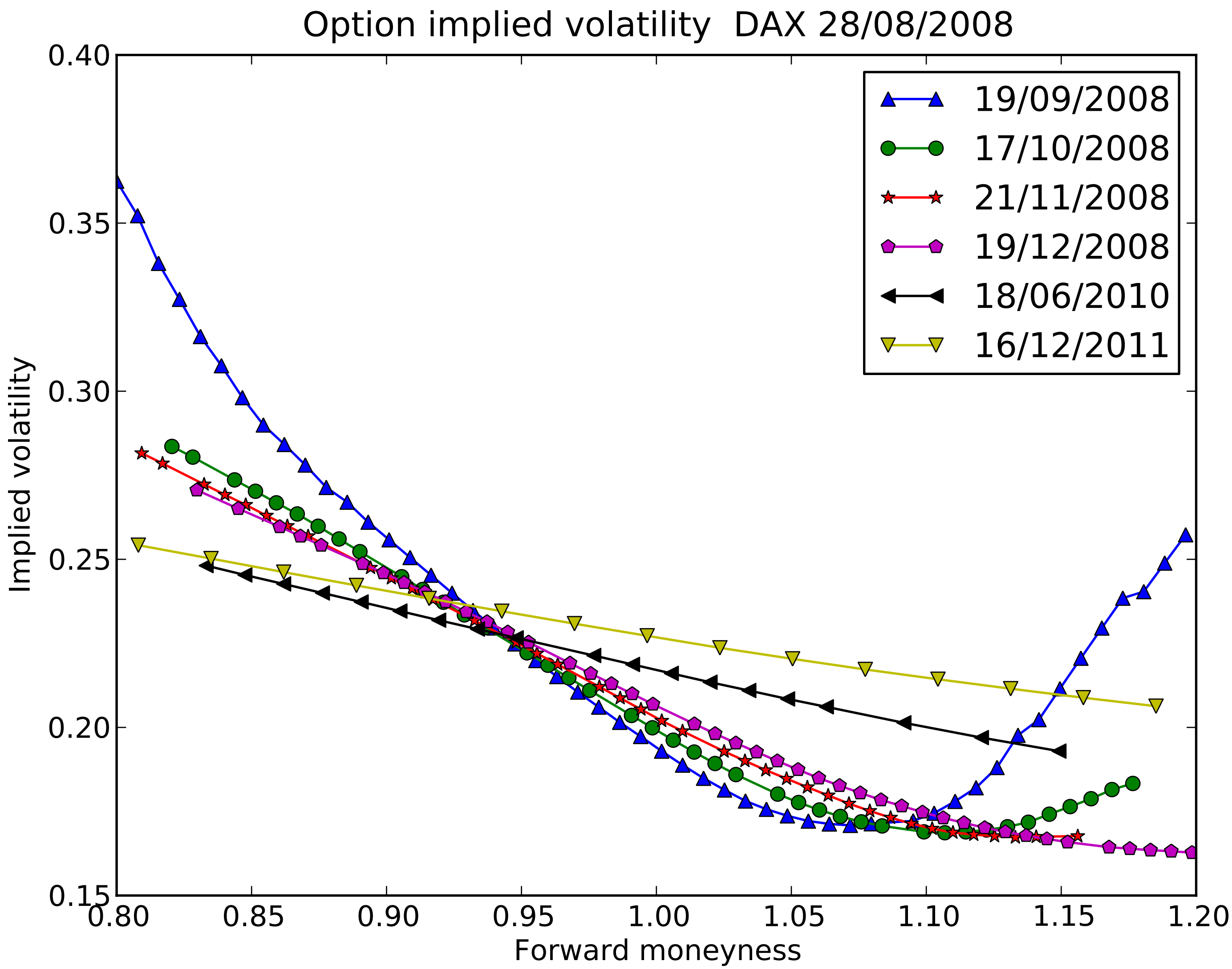
Option implied volatility DAX 28/08/2008
Implied volatility
Forward moneyness
0.40
0.35
0.30
0.25
0.20
0.15
0.80
0.85
0.90
0.95
1.00
1.05
1.10
1.15
1.20
19/09/2008
17/10/2008
21/11/2008
19/12/2008
18/06/2010
16/12/2011

Important facts:

the skew is controlled by ρ

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we have a term structure of skews

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we should have different values for ρ

Why extending the Heston model?

- The dynamics of the implied volatility surface (vanilla options) and the Variance Swap curve are driven by several factors

- On the FX market the skew is stochastic by Carr&Wu

- We have a skew term structure: short term skew $\neq$ long term skew

Double-Heston model

(Christoffensen, Heston, Jacobs 2009)

$$
\begin{aligned}
\frac{ds_t}{s_t} &= \sqrt{v_t^1}\,dz_t^1 + \sqrt{v_t^2}\,dz_t^2 \\
dv_t^1 &= \kappa^1(\theta^1 - v_t^1)dt + \sigma^1\sqrt{v_t^1}\,dw_t^1 \\
dv_t^2 &= \kappa^2(\theta^2 - v_t^2)dt + \sigma^2\sqrt{v_t^2}\,dw_t^2 \\
dz_t^1 dw_t^1 &= \rho^1 dt \\
dz_t^2 dw_t^2 &= \rho^2 dt
\end{aligned}
$$

but

$$
\underbrace{dz_t^1 dz_t^2 = dw_t^1 dw_t^2 = dz_t^1 dw_t^2 = dz_t^2 dw_t^1 = 0}_{AFFINITY}
$$

Recall the Duffie-Filipovic-Schachermayer (2003)'s condition

If $V_t = (v_t^1, v_t^2)^\top$ is a vector affine square root process (thus positive):

$$d\begin{pmatrix} v_t^1 \\ v_t^2 \end{pmatrix} = ...dt + \begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix} d\begin{pmatrix} w_t^1 \\ w_t^2 \end{pmatrix}$$

$$\Downarrow$$

We have strong constraints on the diffusion

$$\Downarrow$$

Strong constraints on the correlation!!

$$\Downarrow$$

We can not correlate v_t^1 and v_t^2 in the Double-Heston

Wishart multi-dim Stochastic Vol

- a dynamics based on the Wishart

$$\frac{ds_t}{s_t} = rdt + Tr\left[\sqrt{\Sigma_t} dZ_t\right]$$

- $d\Sigma_t = (\beta Q^\top Q + M\Sigma_t + \Sigma_t M^\top)dt + \sqrt{\Sigma_t} dW_t Q + Q^\top dW_t^\top \sqrt{\Sigma_t}$
- $Z_t =$ Matrix Brownian Motion correlated with W_t
- $Vol(S_t) = Tr\left[\Sigma_t\right]$ linear combination of the Wishart elements

- $d\Sigma_t = (\beta Q^\top Q + M\Sigma_t + \Sigma_t M^\top)dt + \sqrt{\Sigma_t}dW_t Q + (\sqrt{\Sigma_t}dW_t Q)^\top$
- $\beta Q^\top Q$ with β Gindikin's condition equivalent to Feller.
- M negative definite $\Leftrightarrow$ mean reverting behavior
- Σ_t SYMMETRIC MATRIX SQUARE ROOT PROCESS $(n \times n)$
- Q vol-of-vol.
- $(W_t; t \geq 0)$ is a matrix Brownian motion $(n \times n)$
- the constraint $\beta Q^\top Q$ was relaxed by Cucheiro et al.

Correlation in the Wishart model

- $R \in M_n$ completely describes the correlation structure:

$$
\begin{aligned}
Z_t &= W_t R^\top + B_t \sqrt{\mathbb{I} - RR^\top} \\
&= \text{Matrix Brownian motion!}
\end{aligned}
$$

- This choice is compatible with affinity of the model!!

- Other (few) choices are possible but harder to manage.

- We know how to compute the MGF! (but we need to know how to derive the function).

The Multi-asset model

How to build a multi asset framework:

- Consistent with the smile in vanilla options

- With a general correlation structure

- Analytic as much as possible

Using Heston's model

$$\begin{aligned}
ds_t^1 &= s_t^1 \sqrt{v_t^1}\, dz_t^1 \\
dv_t^1 &= \kappa_1(\theta_1 - v_t^1)dt + \sigma_1 \sqrt{v_t^1}\, dw_t^1 \\
ds_t^2 &= s_t^2 \sqrt{v_t^2}\, dz_t^2 \\
dv_t^2 &= \kappa_2(\theta_2 - v_t^2)dt + \sigma_2 \sqrt{v_t^2}\, dw_t^2
\end{aligned}$$

$dz^1 dz^2 = 0 \;\Leftrightarrow$ Affinity of the model

$$\Downarrow$$

$$\frac{ds^1}{s^1}\frac{ds^2}{s^2} = 0$$

The Wishart Affine Stochastic Correlation model

The model: $s_t = (s_t^1, s_t^2)^\top$ and $\Sigma_t \in M_{(2,2)}$

$$ds_t = diag[s_t]\sqrt{\Sigma_t}dZ_t$$
$$d\Sigma_t = \left(\Omega\Omega^\top + M\Sigma_t + \Sigma_t M^\top\right)dt + \sqrt{\Sigma_t}dW_t Q + Q^\top (dW_t)^\top \sqrt{\Sigma_t}$$

dZ_t is a vector BM (2,1) and dW_t is a matrix BM (2,2):

$$\frac{ds^i}{s^i}\frac{ds^j}{s^j} = \Sigma^{ij}dt$$

Affinity of the infinitesimal generator $\Leftrightarrow dZ_t = dW_t\rho + \sqrt{1 - \rho^\top \rho}dB_t$ where ρ is a vector (2,1) and dB is a vector BM(2,1).

- only 2 parameters to specify the skew
- parsimoniuous model
- the MGF has an exponential affine form and is known

An alternative using Heston's model

$$\begin{aligned}
ds_t^1 &= s_t^1 \sqrt{v_t^1} dz_t^1 \\
dv_t^1 &= \kappa_1(\theta_1 - v_t^1)dt + \sigma_1 \sqrt{v_t^1} dw_t^1 + \sigma_0 \sqrt{v_t^0} dz_t^0 \\
ds_t^2 &= s_t^2 \sqrt{v_t^2} dz_t^2 \\
dv_t^2 &= \kappa_2(\theta_2 - v_t^2)dt + \sigma_2 \sqrt{v_t^2} dw_t^2 + \sigma_0 \sqrt{v_t^0} dz_t^0 \\
dv_t^0 &= \kappa_0(\theta_0 - v_t^0)dt + \sigma_0 \sqrt{v_t^0} dw_t^0
\end{aligned}$$

$$dz^0 dw^0 = \rho dt$$

$$\Downarrow$$

$$ds^1 ds^2 = s^1 s^2 \sigma_0 v^0 \rho dt$$

v^0 is the common factor (but always a constraint due to the affinity constraint).

Recall the FFT to price a Vanilla

$$C(t,T,x) = \frac{1}{2\pi}\int_{-\infty+i\omega_i}^{+\infty+i\omega_i} e^{-i\omega k}\beta(\omega)\Phi_{Y_t}(\tau,\omega)d\omega$$

$$\Phi_{Y_t}(\tau,\omega) = \mathbb{E}_t\left[e^{i\omega Y_{t+\tau}}\right] = e^{\mathrm{Tr}[A(\tau)\Sigma_t]+b(\tau)Y_t+c(\tau)}$$

$$A(\tau) = A_{22}(\tau)^{-1}A_{21}(\tau),$$

with

$$\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) \\ A_{21}(\tau) & A_{22}(\tau) \end{pmatrix} = \exp\tau\begin{pmatrix} M + i\omega Q^\top R^\top & -2Q^\top Q \\ \frac{i\omega(i\omega-1)}{2}\mathbb{I}_n & -\left(M + i\omega Q^\top R^\top\right)^\top \end{pmatrix},$$

A solves a Matrix Riccati equation

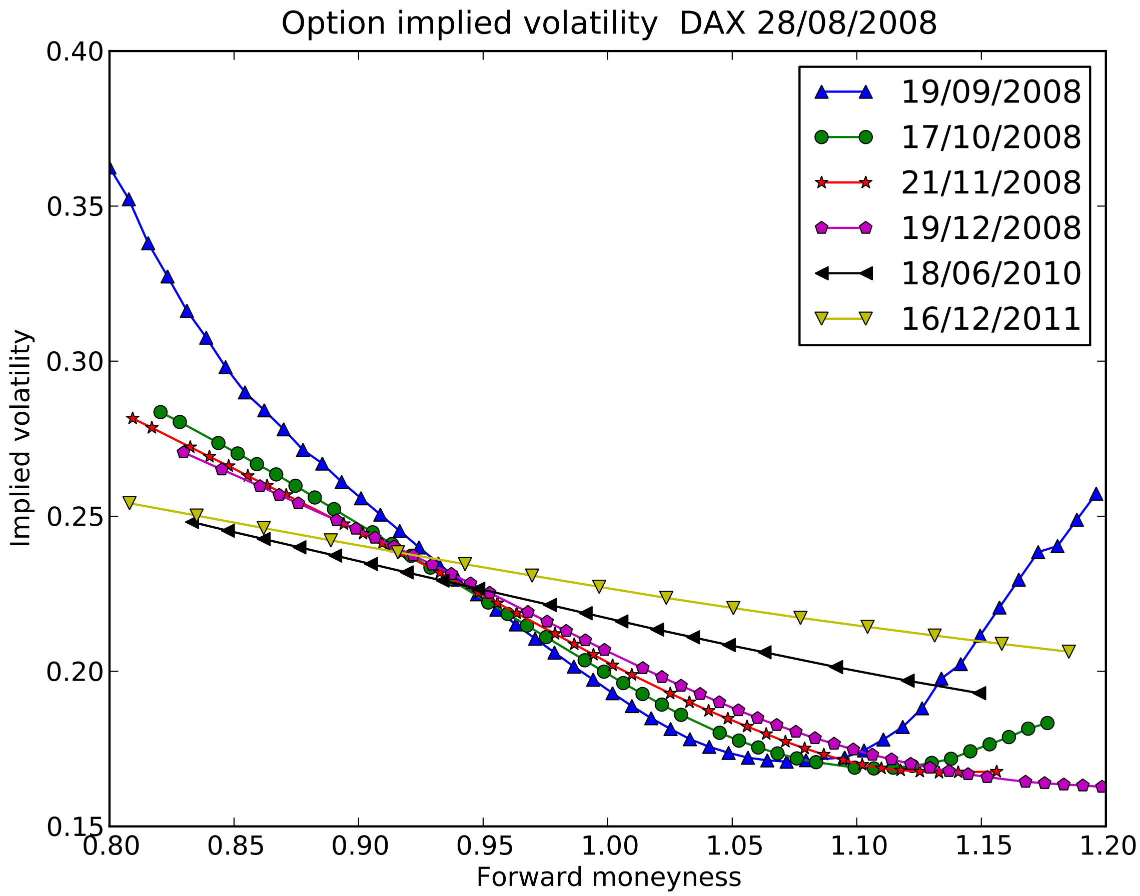
$$\begin{aligned} \partial_\tau A &= A\left(M + i\omega Q^\top R^\top\right) + \left(M + i\omega Q^\top R^\top\right)^\top A + 2AQ^\top QA + \frac{i\omega(i\omega-1)}{2}\mathbb{I} \\ \partial_\tau c &= \mathrm{Tr}[\Omega\Omega^\top A(\tau)] \end{aligned}$$

boundary conditions $A(0) = 0$ and $c(0) = 0$.

Computational remarks

- the fact we know how to compute explicitly the Riccati equations is important when it comes to calibration
- the equation for c supposes that we know how to compute the integral of A
- in some case we need to compute successive integrals of A
- in practice we need to compute the FC 300 times to evaluate one option (or options with same maturity).

Don’t forget the“Smiles”



Short term Implied volatility (σ_{imp}) expansions

For Heston model:

$$\begin{aligned} \sigma_{imp}^2 &= v^0 + \frac{\rho_0 \sigma_0}{2} m_f + \frac{m_f^2 \sigma_0^2}{12 (v^0)^2} \left(4 - 7\rho_0^2\right) \\ m_f &= \log \frac{K}{s_t} \end{aligned}$$

we can read the model parameters directly on the market (computational cost=0) but

- the first order does not allow identification
- the second order leads to concave smile for certain values of ρ!
  $\rho \leq -0.755$ (this property appears in many (most?) of formula of this kind)
- avoiding to calibrate the short term smile leads to strong correlation (absolute value terms). Typical of calibration with norm in price.

Numerical results: the DAX

Parameter	**Heston**	**BiHeston**
κ	1.4078	1.3080
σ	0.9319	1.1202
ρ	-0.5409	-0.3884
θ	0.0838	0.0281
v_0	0.0414	0.0187
κ_1		1.4134
σ_1		0.4822
ρ_1		-0.8395
θ_1		0.0485
v_t^1		0.0229
Error Vol	1.09E-4	7.60E-05

Short term Implied volatility (σ_{imp}) expansions

$$
\begin{aligned}
\sigma_{imp}^2 &= v^0 + \frac{\rho_0\sigma_0}{2} m_f + \frac{m_f^2 \sigma_0^2}{12 (v^0)^2} \left(4 - 7\rho_0^2\right) \\
\sigma_{imp}^2 &= v^0 + v^1 + \frac{\rho_0\sigma_0 v^0 + \rho_1\sigma_1 v^1}{2(v^0+v^1)} m_f \alpha \\
&+ \frac{m_f^2 \alpha^2}{12(v^0+v^1)^2} \left(\sigma_0^2 v^0 + \sigma_1^2 v^1 + 2\left(\rho_0^2\sigma_0^2 v^0 + \rho_1^2 \sigma_1^2 v^1\right) - \frac{15}{4} \frac{\left(\rho_0\sigma_0 v^0 + \rho_1\sigma_1 v^1\right)^2}{(v^0+v^1)} \right),
\end{aligned}
$$

- allows to check the consistency between Heston and BiHeston

$$
\begin{aligned}
\frac{ds_t}{s_t} &= \sqrt{v_t^1} dz_t^1 + \sqrt{v_t^2} dz_t^2 \\
dv_t^1 &= \kappa^1(\theta^1 - v_t^1)dt + \sigma^1 \sqrt{v_t^1} dw_t^1 \\
dv_t^2 &= \kappa^2(\theta^2 - v_t^2)dt + \sigma^2 \sqrt{v_t^2} dw_t^2 \\
dz_t^1 dw_t^1 &= \rho^1 dt \quad dz_t^2 dw_t^2 = \rho^2 dt
\end{aligned}
$$

Short term Implied volatility (σ_{imp}) expansions

$$
\begin{aligned}
\sigma_{imp}^2 &= v^0 + \frac{\rho_0\sigma_0}{2} m_f + \frac{m_f^2\sigma_0^2}{12(v^0)^2}\left(4 - 7\rho_0^2\right) \\
\sigma_{imp}^2 &= \Sigma_t^{11} + \Sigma_t^{22} + \alpha\frac{\mathrm{Tr}\left[RQ\Sigma_t\right]}{\mathrm{Tr}\left[\Sigma_t\right]} m_f \\
&+ \alpha^2\frac{m_f^2}{\left(\mathrm{Tr}\left[\Sigma_t\right]\right)^2}\left(\frac{1}{3}\mathrm{Tr}\left[Q^\top Q\Sigma\right] + \frac{1}{3}\mathrm{Tr}\left[RQ\left(Q^\top R^\top + RQ\right)\Sigma\right] - \frac{5}{4}\frac{\left(\mathrm{Tr}\left[RQ\Sigma\right]\right)^2}{\mathrm{Tr}\left[\Sigma_t\right]}\right),
\end{aligned}
$$

- allows to check the consistency with respect to Heston and BiHeston

$$
\begin{aligned}
\frac{ds_t}{s_t} &= Tr\left[\sqrt{\Sigma_t} dZ_t\right] \\
d\Sigma_t &= (\beta Q^\top Q + M\Sigma_t + \Sigma_t M^\top)dt + \sqrt{\Sigma_t} dW_t Q + Q^\top dW_t^\top \sqrt{\Sigma_t}
\end{aligned}
$$

Short term Implied volatility (σ_{imp}) expansions

$$\sigma_{imp}^2 = v^0 + \frac{\rho_0\sigma_0}{2}m_f + \frac{m_f^2\sigma_0^2}{12(v^0)^2}\left(4 - 7\rho_0^2\right)$$

$$\sigma_{imp}^2 = \Sigma_t^{11} + \alpha m_f(\rho_1 Q_{11} + \rho_2 Q_{21}) + \frac{1}{2}\alpha^2 m_f^2\left[\frac{4(Q_{11}^2 + Q_{21}^2) - 7(\rho_1 Q_{11} + \rho_2 Q_{21})^2}{6\Sigma_t^{11}}\right].$$

- allows to check the consistency with the others

- possible to calibrate a basket model on vanilla options (single stock option)

$$ds_t = diag[s_t]\sqrt{\Sigma_t}dZ_t$$
$$d\Sigma_t = \left(\Omega\Omega^\top + M\Sigma_t + \Sigma_t M^\top\right)dt + \sqrt{\Sigma_t}dW_t Q + Q^\top\left(dW_t\right)^\top\sqrt{\Sigma_t}$$

different models give same shape for the smile, so what?

How the models allow to ”control” the smile

$$
\begin{aligned}
\sigma_{imp}^2 &= v^0 + \frac{\rho_0\sigma_0}{2} m_f \\
\sigma_{imp}^2 &= v^0 + v^1 + \frac{\rho_0\sigma_0 v^0 + \rho_1\sigma_1 v^1}{2(v^0 + v^1)} m_f \alpha \\
\sigma_{imp}^2 &= \Sigma_t^{11} + \Sigma_t^{22} + \alpha \frac{\mathrm{Tr}\left[RQ\Sigma_t\right]}{\mathrm{Tr}\left[\Sigma_t\right]} m_f \\
\sigma_{imp}^2 &= \Sigma_t^{11} + \alpha m_f (\rho_1 Q_{11} + \rho_2 Q_{21})
\end{aligned}
$$

- the natural entry ”points” are v_0 or (v_0^1, v_0^2) or Σ_t etc..
- the practical entry ”points” are ρ etc..

I suppose a shock on the price (.i.e σ_{imp}^2) how it will spread in the model?

Pushing a string

- we find that Feller's condition is not satisfied for the Heston models (many markets)
- we find that Feller's condition is not satisfied for the BiHeston models (many markets). Similar results by Bates (2000)
- Peng & Scaillet obtain similar results (working paper)
- Gindikin is not satisfied for Wishart based models

Parameter	**Heston**	**BiHeston**
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why and what are the consequences?

$$C = E[\phi(s_0 e^{-\frac{1}{2}\int_0^t v_t dt + \int_0^t \sqrt{v_t} dw_t})]$$

Option contains integrated volatility (along the volatility path).

- we can not really distinguish whether the process oscillates a lot or not.
- the maturities are too spaced to sort out if the mean reverting parameter in strong or not.

so the mean reverting will be small, sometimes even equal to zero Bates 2000 and others.

$$2\kappa\theta \geq \sigma^2$$

- the distribution is too close to zero

why and what are the consequences?

$$C = E[\phi(s_0 e^{-\frac{1}{2}\int_0^t v_t^1 dt - \frac{1}{2}\int_0^t v_t^2 dt + \int_0^t \sqrt{v_t^1} dw_t^1 + \int_0^t \sqrt{v_t^2} dw_t^2})]$$

Option contains integrated volatility (along the volatility path).

- things get worse with BiHeston because an option shows the integrated sum of two identical process $\rightarrow$ identification problem.

so the mean reverting will be small, sometimes even equal to zero Bates 2000 and others.

- the distribution is too close to zero

In conclusion:

we use square root process for the volatility because it is positive

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but one of the key parameter that controls the positivity of the process is problematic to estimate from options

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the resulting dynamics is too close to zero

$\Downarrow$

we need positivity by construction.

Back to the beginning

Hull&White's model (1982)

$$
\begin{aligned}
\frac{ds_t}{s_t} &= v_t dw_t^1 \\
dv_t &= v_t \alpha dt + \sigma v_t dw_t^2
\end{aligned}
$$

Scott&Chesney's model (1985)

$$
\begin{aligned}
\frac{ds_t}{s_t} &= e^{v_t} dw_t^1 \\
dv_t &= \kappa(\theta - v_t)dt + \sigma dw_t^2
\end{aligned}
$$

manage the positivity through exponentiation.

Thanks for your attention!